# A NOVEL APPLICATION OF REGGE TRAJECTORIES

C. LOVELACE
*CERN, Geneva*

Received 31 October 1968

We suggest that Dalitz plots for $3\pi$ decays and annihilations are "measurements" of $\pi\pi$ scattering in the double spectral region. Veneziano's formula for crossing-symmetric Regge trajectories both justifies the external mass extrapolation and correctly predicts the experimental results. It also fulfils Adler's self-consistency condition if $\alpha_\rho(m_\pi^2) = \frac{1}{2}$.

Recently Veneziano [1] gave a simple formula which exhibits the Regge pole = resonance duality [2]. We show here how this new development in Regge theory can be applied in a rather unexpected direction, to explain three-particle final state interactions. These then provide direct experimental evidence that Veneziano's formula is not a mere model, but is approximately true in nature. There is also reason, from at least seven parallel predictions, to believe it connected with chiral symmetry.

To start, consider $\pi^+\pi^-$ elastic scattering. There are no $u$ channel resonances, since these would have isospin 2, while the $s$ and $t$ channels are identical. Veneziano's formula [1] then requires exchange-degeneracy between the $f^0$ and $\rho$ trajectories (in agreement with experiment [3]) and gives for the $\pi^+\pi^-$ scattering amplitude

$$A(s,t) = -\beta \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} + \gamma \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} + \dots \tag{1}$$

The $\pi\pi$ amplitudes for the three isostates in the $s$ channel will be

$$\begin{aligned} A^0 &= \tfrac{3}{2}[A(s,t)+A(s,u)] - \tfrac{1}{2}A(t,u) , \\ A^1 &= A(s,t) - A(s,u) , \\ A^2 &= A(t,u) . \end{aligned} \tag{2}$$

$\beta$ in (1) is a constant, and its coefficient contains, besides the $\rho$ and $f^0$ trajectories, unit-spaced daughters of alternating isospin. The $\gamma$ term contains the daughters only (and might cancel them), while further terms start with the second daughter. Veneziano's formula is only strictly valid when the trajectories are exactly linear, so that resonances are approximated by poles [4].

Now consider what happens when we take one of the external pions off its mass-shell. Comparison with Feynman diagrams shows that the poles of (1) must be in the variables $s = (k_1+k_2)^2$, $t = (k_1 - k_1')^2$, i.e., that the Regge trajectories cannot depend kinematically on the external masses. Only $\beta$, $\gamma$ etc. can therefore vary as we go off-shell. Adler's self-consistency condition [5] (recall that this is a dynamical constraint, not true for individual Feynman graphs) requires (1) to vanish when $s = t = u = \mu^2$ and one of the pions has zero mass. Here $\mu$ is the physical pion mass. The first term of (1) contains a factor

$$\alpha(s) + \alpha(t) - 1 \tag{3}$$

which will vanish there provided

$$\alpha(\mu^2) = \tfrac{1}{2} . \tag{4}$$

If we take the $\rho$ trajectory as linear, and the $\rho$ mass as 764 MeV, this gives $\alpha_\rho(0) = 0.483$ MeV. Now $\alpha_\rho(s)$ must deviate from linearity because of unitarity corrections, and we can estimate the size of these by comparing different determinations of $\alpha_\rho(0)$ from Chew-Frautschi plots and Regge fits. Our table shows that (4) is true to within the accuracy to which we expect the Veneziano formula itself to hold. Thus the Veneziano formula *with one term only* [$\gamma$, etc. = 0 in eq. (1)] and the experimental trajectory predicts Adler's self-consistency condition. (We shall denote such parallelisms, when a chiral symmetry prediction is also a Veneziano prediction, by roman numerals. This is I.) Either this is coincidence and the higher terms just happen to cancel at this point, or else there really is only one Veneziano term present in the $\pi\pi$ case. We shall assume the latter.

The same Regge trajectories will occur wher-

Table 1.
Old and new determinations of the $\rho$ intercept

| | Method | $\alpha_\rho(0)$ |
|---|---|---|
| Old | Regge fit to charge-exchange [21] | 0.57 ± 0.01 |
| | Finite energy sum rule [3] | 0.55 |
| | Linear interpolation between the dip and the $\rho$ | 0.485 |
| | Linear extrapolation from the $\rho$ and g | 0.462 |
| New | Soft pion theorem | 0.483 |
| | Fit to $\eta$ decay (data of ref. 17) | 0.491 ± 0.002 |
| | Fit to $\tau$ decay (data of ref. 19) | 0.528 ± 0.006 |

ever all four external particles have the quantum number of the pion. If one is a zero mass pion, then Adler's self-consistency condition must again hold. To get the required zeros from the Veneziano formula, all actual particles with the pion's quantum numbers must satisfy

$$\alpha_\rho(m^2) = \tfrac{1}{2} + n\,, \qquad n = 0, 1, 2, \ldots \qquad (5)$$

This is reasonable, since the daughters of the pion's Regge recurrence would be just here for parallel trajectories. Adler's self-consistency conditions thus requires quantization of the external masses, which is strange viewed from conventional theory, but by no means unphysical. If a particle such as $\eta$, whose mass does not coincide with a pion recurrence, decays into three pions through electromagnetic or weak interactions, then Veneziano's formula predicts that only one of the three soft pion zeros [6] will occur. This is experimentally correct, as we shall see.

The linear trajectory which gives eq. (4) and the $\rho$ mass is

$$\alpha(s) = 0.483 + 0.885\,s \qquad (6)$$

If we now expand the first term of eq. (1) as a sum of poles

$$A(s,t) = \beta[\alpha(s) + \alpha(t) - 1] \times \sum_{n=0}^{\infty} \binom{\alpha(t) + n - 1}{n} (n + 1 - \alpha(s))^{-1} \qquad (7)$$

and decompose the residues into partial waves, we obtain the relative $\pi\pi$ widths of the various daughters:

$$\rho \text{ daughter widths} \quad P_1 : S_0 = 2 : 9 \qquad (8a)$$

$$\mathrm{f}^0 \text{ daugther widths} \quad D_0 : P_1 : S_0 = 9 : 10 : 0 \qquad (8b)$$

[The round numbers would be exact if $\mu = 0$.] Chiral symmetry also predicts [7] equality of the $\rho$ and $\epsilon$ masses (II) and the width relation (8a) (III). The latter is a Veneziano prediction only if $\gamma = 0$ in eq. (1). Substituting eq. (6) and the first term of eq. (1) into eq. (2) gives the $\pi\pi$ scattering lengths

$$a_0 = 0.395\beta, \quad a_2 = -0.103\beta\,. \qquad (9)$$

The ratio is within 10% of Weinberg's value [8] (IV).

To proceed further we must give the resonances finite width by adding an imaginary part to $\alpha(s)$ for $s > 4\mu^2$. Unfortunately there is no consistent way of doing this. The resonances at the same mass will all acquire the same total width by eq. (7), but must have different elastic widths by eq. (8), which contradicts elastic unitarity. Also the boundary of the double spectral region will be at $s = 4\mu^2$, $t = 4\mu^2$ instead of the correct curve, so that parallel trajectories at threshold violate the centrifugal barrier [9]. Furthermore if $\alpha(t)$ is not a polynomial, eq. (7) shows that the $s$ channel resonances will have high-spin "ancestors" as well as daughters. For the $\rho$ only, this last difficulty can be avoided by not giving any imaginary part to the first zero, i.e., replacing eq. (1) by

$$A(s,t) = [0.885(s,t) - 0.034] \times \beta\, \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}\,. \qquad (10)$$

To apply Veneziano's marvellous formula to experiment, we must therefore claim poetic licence to adjust $\mathrm{Im}\,\alpha$ to the needs of the moment. For our first application the $\epsilon$ width will be more important than the $\rho$, so we take

$$\alpha(s) = 0.483 + 0.885s + \mathrm{i}\,0.28\sqrt{s - 4\mu^2} \qquad (11)$$

which, when eq. (10) is analysed into partial waves, corresponds to an $\epsilon$ 280 MeV broad. Experimentally [10] it seems to be between 250 and 500 MeV.

Now we continue one of the external pions to a large positive mass, i.e., we consider the $3\pi$ decay of a particle with the same quantum numbers as the pion. This will have the same Regge trajectories as $\pi\pi$ scattering so the Veneziano formula can only differ in the values of $\beta, \gamma$, etc. in eq. (1). With a single Veneziano term, we can then predict everything except the total decay rate. The Dalitz plot densities for various charge combinations should be

$$[X^\pm \to \pi^\pm\pi^\pm\pi^\mp] = N|A(s,t)|^2\,, \qquad (12a)$$

$$[X^{\pm} \to \pi^{\pm}\pi^0\pi^0] = [X^0 \to \pi^+\pi^-\pi^0] = \tag{12b}$$
$$= \tfrac{1}{4}N|A(t,u) - A(s,t) - A(s,u)|^2 ,$$
$$[X^0 \to 3\pi^0] = \tfrac{1}{4}N|A(s,t) + A(s,u) + A(t,u)|^2 . \tag{12c}$$

Here $N$ is a normalization constant. $s$, $t$, $u$ are the squares of the masses of the final dipion combinations (symmetry shows which). If the decay process couples to all $\pi\pi$ resonances, the $A(s,t)$ is given by eqs. (10) and (11).

Thus, if the Veneziano formula realy describes the interference of $s$ and $t$ channel Regge trajectories, then it must necessarily also describe how overlapping resonances interfere in three-particle decays. Traditionally, overlapping resonances in isobar models have simply been added in the matrix element, though the correctness of this has been doubted [11]. According to the Veneziano formula, duality should occur, expressed by the equality of the $s$ and $t$ channel forms of eq. (7). We shall now test this experimentally.

In addition to the classic $K \to 3\pi$ and $\eta \to 3\pi$ decays, it is interesting to consider $\bar{p}n \to \pi^+\pi^-\pi^-$ at rest, because the initial state is known experimentally [12] to be entirely $^1S_0$, as we might expect at threshold. This has the quantum numbers of the pion. No $\rho$ is observed in the $3\pi$ Dalitz plot [12]. We therefore have to remove the leading trajectory from the Veneziano formula by crossing out the first factor of eq. (10). This decoupling of the $\rho$ is a purely phenomenological assumption, but *any* theory of final state interactions applied to this decay is forced to make it, as there is nothing in the final state to forbid $\rho$. Taking a single Veneziano term in eq. (12a), we then have a $1\frac{1}{2}$ parameter fit, since the normalization is not predicted and the $\epsilon$ width is only roughly known from other experiments. The 0.28 in eq. (11) was in fact obtained from this fit. The resulting Dalitz plot is shown in fig. 1. The agreement would probably be still better with a more complicated $\mathrm{Im}\,\alpha(s)$ to make the $\epsilon$ broader below than above (in accord with other evidence [10]), but it is already remarkably good for so few parameters. This strange Dalitz plot is just what to expect from the Veneziano formula. The criss-cross of vertical and horizontal poles, with diagonal zeros to cancel the double poles at their intersections [4], is clearly visible. The hole in the middle comes from the zero at $\alpha(s) + \alpha(t) = 3$, and thus confirms that this is not filled in by secondary Veneziano terms. Even more spectacular is the enhancement at the upper right edge. Any naive isobar model must attribute this [13] to a strong

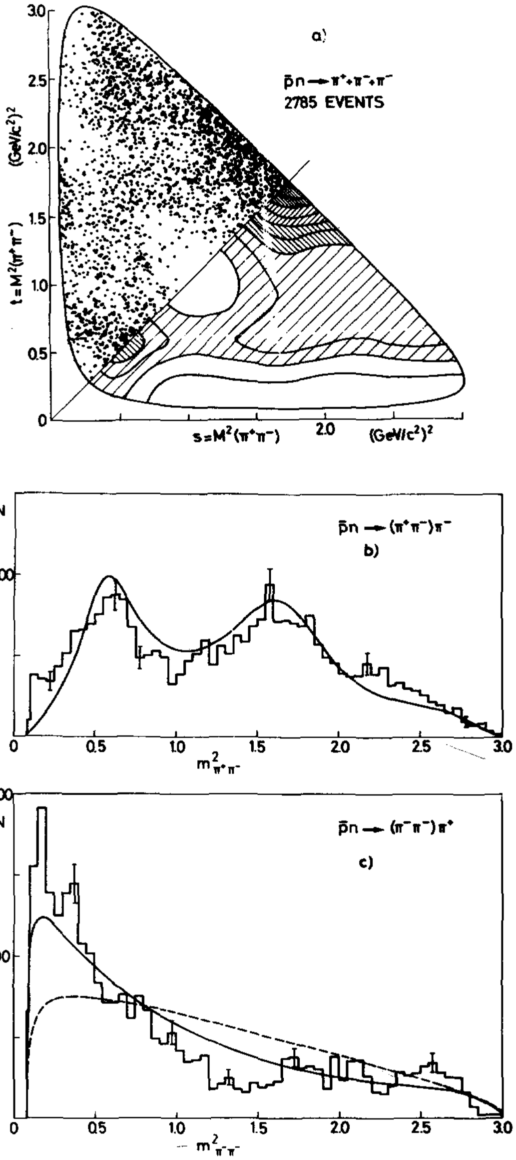


Fig. 1. Dalitz plot and its projections for $\bar{p}n \to \pi^-\pi^-\pi^+$ at rest (data of ref. 12). The solid curves and contours are from the Veneziano formula with the trajectory (11), the dashed curve is phase space.

low energy $\pi^-\pi^-$ interaction in the isospin 2 state - a most unpalatable hypothesis. Here it corresponds to the region where $s$ and $t$ are both large and positive, and demonstrates the blow-up of the double spectral function in this direction, for a theory with rising Regge trajectories According to the Veneziano formula, this Dalitz plot is a window through which we can observe $\pi\pi$ scattering in the double spectral region. Furthermore, as the $\rho$ decouples, Veneziano's formula says the $f^0$ will also decouple, so the second bump in fig. 1b must be the long sought $\rho'$. Since eq. (11) evidently fits, we can use it to determine the $\rho'$ mass and width as 1310 and 400 MeV, which seem reasonable values for the electromagnetic form factors [14].

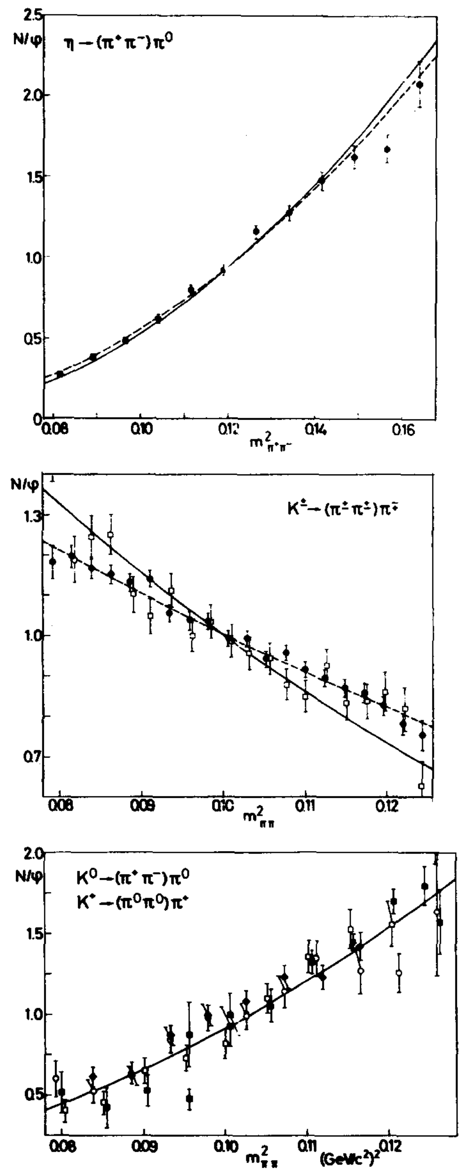


Fig. 2. Spectra (divided by phase space) for the bracketed $\pi\pi$ combinations in various $\eta$ and K decays (data of refs. 17-19). The solid curves are zero-parameter predictions using eq. (11), the dashed curves are one-parameter fits relaxing the soft pion constraint (see table 1).

Now we consider $\eta$ and $\tau$ decay, which behave according to prevalent theories [e.g. 15] as if the initial particle had the quantum numbers of the pion. Here we do not of course decouple the $\rho$, since the first factor of eq. (10) will automatically give the zero required by soft pion theory for $\tau$ decay [16] (V), and one of its zeros [6] for $\eta$ decay (VI). As we explained above [eq. (5)] there is only one zero because the $\eta$ mass does not coincide with a pion recurrence. No arbitrary assumption about linear matrix elements is now needed. Various $\pi\pi$ spectra divided by phase space are shown in fig. 2. (We did not attempt pion mass-difference corrections.) The solid curves are zero-parameter predictions (since the total decay rate does not enter), obtained by substituting eqs. (10) and (11) into eqs. (12a) and (12b). For $\eta$ decay [17] the agreement is obviously excellent, but for $\tau$ decay though most of the experiments [18] fit, the largest [19] if taken literally (this is preliminary data uncorrected for systematic errors) suggests some discrepancy. The dashed lines in fig. 2 are one-parameter fits in which the soft pion constraint eq. (4) was relaxed. The resulting $\alpha_\rho(0)$ are shown in table 1. We see that the spread in the new determinations is less than that in the old ones.

It seems to us that these experimental successes are a good reason for taking the Veneziano formula seriously, not as a model, but as an approximate theory of the strong interactions. The quantity of previous papers on the subject shows how non-trivial it is to account for the $\tau$ and $\eta$ decay final state interactions without conflicting with other $\pi\pi$ evidence. Unless we believe in a low mass isospin $2\pi\pi$ resonance or virtual bound state, $\bar{p}n \to 3\pi$ has no other explanation at all. With appropriate trajectories our theory can be applied wherever three particles occur with definite total spin. We emphasize that $\mathrm{Im}\,\alpha(s)$ should be chosen in each problem to give the widths of the particular contributing resonances - there can be no universally valid formula for it.

Perhaps the most exciting prospect, however, is of unifying Regge theory and current algebra. More parallelisms and convergent results can easily be discovered. Thus, since the Veneziano formula extrapolates correctly from $m_\pi$ to $2m_N$, it should convey us safely from 0 to $m_K$. We then get for all soft meson theorems to hold in $\pi$K scattering, the obvious SU(3) extension of eq. (4)

$$\alpha_{K^*}(m_K^2) = \tfrac{1}{2}\,, \tag{13}$$

predicting for parallel trajectories

$$m_{K*}^2 - m_K^2 = m^2 - m_\pi^2 \qquad (14)$$

which is an SU(6) result and true, though the derivation actually contradicts SU(6) because eq. (13) does not allow vector and pseudoscalar mesons to be degenerate. Again, if the pion conspirator trajectory is parallel to the $\rho$, eq. (4) gives

$$m_{A1}^2 - m_\rho^2 = m_\rho^2 - m_\pi^2 \qquad (15)$$

which is experimentally exact and agrees for $m_\pi = 0$ with a deduction by Weinberg [20] from the algebra of fields (VII). We have now counted seven such coincident predictions, which leads us to conjecture:

Chiral symmetry for soft mesons + absence of exotic resonances = Veneziano formula with no secondary terms.

The importance of verifying this can hardly be overestimated - it would virtually amount to a complete theory of the strong interactions.

I am grateful to Drs. C. Schmid, M. Jacob, R. G. Roberts and F. Wagner for discussions.

*References*


1. G.Veneziano, Nuovo Cimento 57A (1968) 190.
2. R.Doolen, D.Horn and C.Schmid, Phys. Rev. 166 (1968) 1768;
   C.Schmid, Phys. Rev. Letters 20 (1968) 689.
3. V.Barger and R.J.N.Phillips, Evidence for a flat Pomeranchuk trajectory and location of a secondary rho Regge pole, Vienna Conference (1968) paper 51.
4. C.Goebel, unpublished.
5. S.Adler, Phys. Rev. 137 (1965) B1022.
6. D.Sutherland, Phys. Letters 23 (1966) 384; Nucl. Phys. B2 (1967) 433;
   J.S.Bell and D.Sutherland, Nucl. Phys. 4B (1968) 315.
7. F.Gilman and H.Harari, Phys. Rev. 165 (1968) 1803;
   S.Weinberg, Algebraic realizations of chiral symmetry, M.I.T. preprint (1968).
8. S.Weinberg, Phys. Rev. Letters 17 (1966) 616.
9. A.Barut and D.Zwaniger, Phys. Rev. 127 (1962) 974.
10. C.Lovelace, R.M.Heinz and A.Donnachie, Phys. Letters 22 (1966) 332;
    A.B.Clegg, Phys. Rev. 163 (1967) 1664;
    E.Malamud and P.E.Schlein, Phys. Rev. Letters 19 (1967) 1056;
    C.Lovelace, Proc. Heidelberg Conf. (1967) p.79 and Proc. Irvine Conf. (1967) to be published.
11. R.D.Amado, Phys. Rev. 158 (1967) 1414.
12. P.Anninos, L.Gray, P.Hagerty, T.Kalogeropoulos, S.Zenone, R.Bizzarri, G.Ciapeti, M.Gaspero, I.Laakso, S.Lichtman and G.C.Moneti, Phys. Rev. Letters 20 (1968) 402.
13. J.Clayton, P.Mason, H.Muirhead, P.Renton, R. Rigopoulos, E.Simopoulou, P.Tsilimigras and A.Vayaki-Serafimidou, Pion-pion correlations at low energies, Vienne Conference (1968) paper 276.
14. G.Höhler, R.Strauss and H.Wunder, Prediction of the isovector nucleon form factor from $\pi$N scattering and pion form factor data, Karlsruhe preprint (1968).
15. N.Cabibbo, Proc. Berkeley Conf. (1966) p. 29;
    G.Trilling, Proc. Argonne Conf. on Weak interactions (1966) p. 115;
    and for a contrary view M.Veltman and J.Yellin, Phys. Rev. 154 (1967) 1469.
16. M.Suzuki, Phys. Rev. 144 (1966) 1154;
    C.Callan and S.Treiman, Phys. Rev. Letters 16 (1966) 153;
    Y.Hara and Y.Nambu, Phys. Rev. Letters 16 (1966) 875;
    D.K.Elias and J.C.Taylor, Nuovo Cimento 44A (1966) 518; 48A (1967) 814;
    H.Abarbanel, Phys. Rev. 153 (1967) 1547.
17. A.Cnops, G.Finocchiaro, P.Mittner, J.Dufey, B.Gobbi, M.Pouchon and A.Müller, Phys. Letters 27B (1968) 113.
18. G.Kalmus, A.Kernan, R.Pu, W.Powell and R. Dowd, Phys. Rev. Letters 13 (1964) 99;
    V.Bisi, G.Borreani, R.Cester, A.De Marco-Trabucco, M.Ferrero, C.Garelli, A.Marzari Chiesa, B.Quassiati, G.Rinaudo, M.Vigogne and A.Werbrouck, Nuovo Cimento 35 (1965) 768;
    B.Nefkens, A.Abashian, R.Abrams, D.Carpenter, G.Fisher and J.H.Smith, Phys. Rev. 157 (1967) 1233;
    H.Hopkins, T.Bacon and F.Eisler, Phys. Rev. Letters 19 (1967) 185;
    W.Butler, R.Bland, G.Goldhaber, S.Goldhaber, A.Hirata, T.O'Halloran, G.Trilling and C.Wohl, A study of 10000 $\tau^+$ decays, UCRL-18420 and Vienna Conf. (1968) paper 940.
19. T.Mast, L.Gershwin, M.Alston-Garnjost, R.Bangerter, A.Barbaro-Galtieri, J.Murray, F.Solmitz, R.Tripp and B.Webber, 40000 tau decays, UCRL-18329 and Vienna Conf. (1968) paper 670.
20. S.Weinberg, Phys. Rev. Letters 18 (1967) 507.
21. G.Höhler, J.Baacke, H.Schaile and P.Sonderegger, Phys. Letters 20 (1966) 79.


* * * * *